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Markov Properties of the Fluctuation Limit of a Particle System with Death*

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Consider a sequence of independent Brownian motions in R^d whose initial positions are distributed according to a probability measure μ on $R^d \times R_+$ and disappear after an exponentially distributed lifetime with parameter λ. It is shown that the fluctuation process of the empirical distribution of the first n motions converges weakly in the Skorokhod spaces $D([0, T_0), \mathscr{S}'(R^d))$ and $D([T_0, \infty), \mathscr{S}'(R^d))$ as $n \to \infty$, where $T_0 = \inf\{t : \mu(R^d \times [t, \infty)) = 0\}$, to a continuous, centered Gaussian process X. Conditions for X to be Markovian are determined in terms of μ. The cases $\lambda > 0$ and $\lambda = 0$ (i.e., the motions go on forever) exhibit different Markovian properties. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let $B^k = \{B_t^k, t \geqslant 0\}$, $k = 1, 2, \ldots$, be independent Brownian motions in R^d, each one starting from a point distributed according to a probability measure μ on $R^d \times R_+$, and disappearing after an exponentially distributed lifetime with parameter λ. In this paper we study the asymptotic behavior of the fluctuation process of the empirical distribution of the first n motions as $n \to \infty$, and the Markov property of the fluctuation limit process. The case $\lambda = 0$ (i.e., the motions go on forever) was considered in [6], and the case $\lambda = 0$ and $\mu(R^d \times \{0\}) = 1$ was studied by Itô [7]. In contrast with the case $\lambda = 0$, where the fluctuation limit process is Markovian, in the model with death the Markov property does not hold in general.

The processes we consider can be regarded as $\mathscr{S}'(R^d)$-valued, where $\mathscr{S}'(R^d)$ is the Schwartz space of tempered distributions, i.e., the topological dual of the space $\mathscr{S}(R^d)$ of rapidly decreasing infinitely differentiable real functions on R^d. The duality between these two spaces will be denoted by $\langle \cdot , \cdot \rangle$.

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We will prove that the fluctuation processes of the empirical distribution of the first n motions converge weakly in the Skorokhod spaces $D([0, T_0), \mathscr{S}'(R^d))$ and $D([T_0, \infty), \mathscr{S}'(R^d))$ as $n \to \infty$, where $T_0 = \inf\{t: \mu(R^d \times [t, \infty)) = 0\}$, to a continuous, centered Gaussian process X, and that X has the Markov property in an interval $[a, b]$ if and only if $\lambda = 0$ or $\mu(R^d \times (a, b]) = 0$. Thus the Markov property does not hold for X in $[a, b]$ if and only if births and deaths are both possible in $(a, b]$. This result should be considered natural since it is in agreement with the fact that the same Markovian behavior holds for a single Brownian motion with initial distribution μ and exponential lifetime. We will also prove that if $\mu(R^d \times (a, b]) = 0$ then the process X has the *strict Markov property* (defined below) on $(a, b]$, and we will obtain the Langevin equation for X on $(a, b]$. In particular the process $\{X_{T_0+t}, t \geqslant 0\}$ is strictly Markovian.

We will now recall the Markov property and the *strict Markov Property* for Gaussian $\mathscr{S}'(R^d)$-valued processes (see [6]).

The markov property of an $\mathscr{S}'(R^d)$-valued continuous, centered, Gaussian process X is equivalent to the statement that given $s < t$ there exists an $\mathscr{S}'(R^d)$-valued random variable $X_{s,t}$ such that for each $\psi \in \mathscr{S}(R^d)$, $\langle X_{s,t}, \psi \rangle$ belongs to $\langle X_s, \mathscr{S}(R^d) \rangle^-$ (the L^2-closure of the linear space of random variables $\{\langle X_s, \phi \rangle, \phi \in \mathscr{S}(R^d)\}$), and

$$\operatorname{Cov}(\langle X_r, \phi \rangle, \langle X_t, \psi \rangle) = \operatorname{Cov}(\langle X_r, \phi \rangle, \langle X_{s,t}, \psi \rangle) \tag{M}$$

for all $r \leqslant s$ and $\psi, \phi \in \mathscr{S}(R^d)$. We define the *strict Markov property* of a centered, Gaussian $\mathscr{S}'(R^d)$-valued process as follows: given $s < t$ there exists a continuous linear mapping $\mathscr{U}_{s,t}: \mathscr{S}(R^d) \to \mathscr{S}(R^d)$ such that

$$\operatorname{Cov}(\langle X_r, \phi \rangle, \langle X_t, \psi \rangle) = \operatorname{Cov}(\langle X_r, \phi \rangle, \langle X_s, \mathscr{U}_{s,t}\psi \rangle) \tag{SM}$$

for all $r \leqslant s$ and $\psi, \phi \in \mathscr{S}(R^d)$. Clearly this is a special case of the Markov property since $\langle X_s, \mathscr{U}_{s,t}\phi \rangle$ belongs to $\langle X_s, \mathscr{S}(R^d) \rangle$. If $\{\mathscr{U}_{s,t}, s \leqslant t\}$ is a reverse evolution system on $\mathscr{S}(R^d)$, i.e., $\mathscr{U}_{t,t} = I$ and $\mathscr{U}_{r,s}\mathscr{U}_{s,t} = \mathscr{U}_{r,t}$ for $r \leqslant s \leqslant t$, then it is easy to see that (SM) is equivalent to

$$\operatorname{Cov}(\langle X_s, \phi \rangle, \langle X_t, \psi \rangle) = \operatorname{Cov}(\langle X_s, \phi \rangle, \langle X_s, \mathscr{U}_{s,t}\psi \rangle). \tag{SM'}$$

The topologies on $\mathscr{S}(R^d)$ and $\mathscr{S}'(R^d)$ are well known (see [6]), and we will denote by $\mathscr{S}(R^d) \oplus R$ the direct sum of $\mathscr{S}(R^d)$ and the space of constant functions on R^d with the direct sum topology.

The space of right-continuous with left limits functions from $[a, b)$ into $\mathscr{S}'(R^d)$ is denoted by $D([a, b), \mathscr{S}'(R^d))$ and is endowed with a Skorokhod-type topology [9].

The following notation will be used throughout:

$$p_t(x, y) = (2\pi t)^{-d/2} \exp\{-|y-x|^2/2t\}, \qquad t > 0,\ x,\ y \in R^d.$$

$$\mathscr{T}_t\phi(x) = \int_{R^d} \phi(y)\, p_t(x, y)\, dy, \qquad t > 0,\ \mathscr{T}_0 = I \text{ (the Brownian semigroup)}.$$

For $s \leqslant t$,

$$\mu_{st}(x) = \int_{R^d \times [0,s] \times [t,\infty]} p_{t-u}(x, y)\, \nu(dy, du, dl),$$

$$f_{st}(\phi) = \int_{R^d \times [0,s] \times [t,\infty]} \phi(x)\, \nu(dx, du, dl),$$

where

$$\nu(dx, dr, dl) = \begin{cases} \lambda \exp\{-\lambda(l-r)\}\, dl\mu(dx, dr), & l \geqslant r, \\ 0, & l < r, \end{cases}$$

if $\lambda > 0$ and

$$\nu(dx, dr, dl) = \delta_{+\infty}(dl)\, \mu(dx, dr)$$

if $\lambda = 0$, where $\delta_{+\infty}$ is the Dirac measure concentrated at $+\infty$. Observe that for $\lambda = 0$ the above expressions satisfy

$$\mu_{st}(x) = \int_{R^d \times [0,s]} p_{t-u}(x, y)\, \mu(dy, du), \qquad s \leqslant t,$$

and

$$f_{st}(\phi) = \int_{R^d \times [0,s]} \phi(x)\, \mu(dx, du), \qquad s \leqslant t.$$

Then, for $\lambda = 0$ we have $\mu_{st}(x) \equiv \mu_s(x)$ and $f_{st}(\phi) \equiv f_s(\phi)$, where the right-hand sides are the notation used in [6].

The definitions above are valid for bounded, measurable, real functions ϕ on R^d.

Let $T_0 = \inf\{t : \mu(R^d \times [t, \infty)) = 0\}$. We will assume that the measure μ is sufficiently smooth so that for each $T < T_0$,

$$\int_{R^d \times (s,t]} e^{-\lambda(t-u)}\, \mu(dx, du) \leqslant K(t-s)$$

for all $s < t \leqslant T$ and some constant K depending on T.

2. RESULTS AND PROOFS

Let $W \equiv \{W_t, t \geq 0\}$ be a standard Brownian motion on R^d (starting from 0), and (Z, τ, θ) and $R^d \times R_+ \times R_+$-valued random variable with distribution $\nu(dx, dr, dl)$ independent of W. Let W^k, (Z^k, τ^k, θ^k), $k = 1, 2, \ldots$, be independent copies of W, (Z, τ, θ). We consider as state space $R^d \cup \{\gamma\}$, where γ is an external point, and define the kth Brownian motion with distribution μ and exponential lifetime (i.e., beginning at time τ^k and position Z^k and dying at time θ^k) as

$$B_t^k = \begin{cases} \gamma, & \text{if } \; t < \tau^k \text{ or } t \geq \theta^k, \\ Z^k + W_{t-\tau^k}^k, & \text{if } \; \tau^k \leq t < \theta^k. \end{cases}$$

Let N_t^n denote the empirical distribution at time t of the first n processes, i.e.,

$$N_t^n = \sum_{k=1}^{n} \delta_{B_t^k}, \qquad t \geq 0,$$

where δ_x is the Dirac measure concentrated at x. $N^n \equiv \{N_t^n, t \geq 0\}$ can be regarded as an $\mathscr{S}'(R^d)$-valued process given by

$$\langle N_t^n, \phi \rangle = \sum_{k=1}^{n} \phi(B_t^k), \qquad \phi \in \mathscr{S}(R^d), \tag{2.1}$$

where the definition of ϕ is extended to γ by $\phi(\gamma) = 0$.

We define the fluctuation process $X^n \equiv \{X_t^n, t \geq 0\}$ by centering and normalizing

$$\langle X_t^n, \phi \rangle = n^{-1/2} \sum_{k=1}^{n} [\phi(B_t^k) - E\phi(B_t^k)], \qquad \phi \in \mathscr{S}(R^d). \tag{2.2}$$

Note that (2.1) and (2.2) are also well defined for $\phi \in \mathscr{S}(R^d) \oplus R$.

LEMMA 2.1. *For each $t \geq 0$ and $\phi \in \mathscr{S}(R^d) \oplus R$,*

$$E\langle N_t^n, \phi \rangle = n \int_{R^d} \phi(y)\, \mu_{tt}(y)\, dy, \tag{2.3}$$

and for $s \leq r \leq t$, ϕ, $\psi \in \mathscr{S}(R^d) \oplus R$,

$$\begin{aligned}
&\operatorname{Cov}(\langle N_s^n, \phi\rangle, \langle N_t^n, \psi\rangle) \\
&= n\left\{\int_{R^d} \phi(z)\,\mu_{ss}(z)\,e^{-\lambda(t-s)}\mathcal{T}_{t-s}\psi(z)\,dz\right. \\
&\quad -\int_{R^d} \phi(z)\,\mu_{ss}(z)\,dz\left[\int_{R^d} \mu_{ss}(z)\,e^{-\lambda(t-s)}\mathcal{T}_{t-s}\psi(z)\,dz\right. \\
&\quad \left.\left.+\int_{R^d\times(s,t]} e^{-\lambda(t-r)}\mathcal{T}_{t-r}\psi(x)\,\mu(dx,dr)\right]\right\}. \qquad (2.4)
\end{aligned}$$

Proof. By (2.1),

$$E\langle N_t^n, \phi\rangle = \sum_{k=1}^{n} E\phi(B_t^k).$$

Conditioning with respect to (Z^k, τ^k, θ^k) and using Fubini's theorem we have

$$\begin{aligned}
E\phi(B_t^k) &= \int_{R^d\times[0,t]\times[t,\infty)} \mathcal{T}_{t-r}\phi(x)\,\nu(dx, dr, dl) \\
&= \int_{R^d} \phi(y)\,\mu_{tt}(y)\,dy, \qquad k = 1, 2, ...,
\end{aligned}$$

which proves (2.3).

Using the independence of the B^k, $k \geqslant 1$, a similar calculation yields (2.4).

We will now state and prove the fluctuation limit theorem.

THEOREM 2.1. *There exists a continuous centered Gaussian $\mathcal{S}'(R^d)$-valued process $X \equiv \{\langle X_t, \phi\rangle,\ t \geqslant 0,\ \phi \in \mathcal{S}(R^d)\}$ with covariance functional given by the right-hand side of* (2.4) *with $n = 1$, and X^n converges weakly as $n \to \infty$ to X in $D([0, T_0), \mathcal{S}'(R^d))$ and $D([T_0, \infty), \mathcal{S}'(R^d))$.*

Proof. The stated process X exists by Kolmogorov's extension theorem for $\mathcal{S}'(R^d)$-valued processes.

By Theorem 1 and Example 1 of [8], to prove that X has a continuous version it suffices to show that for each $\phi \in \mathcal{S}(R^d)$ the real Gaussian process $\{\langle X_t, \phi\rangle, t \geqslant 0\}$ satisfies the Dudley–Fernique condition. Since the covariance functional of X is given by the right-hand side of (2.4) with $n = 1$, for each $\phi \in \mathcal{S}(R^d)$ and $s \leqslant t$ we have

$$
\begin{aligned}
&E[\langle X_t, \phi\rangle - \langle X_s, \phi\rangle]^2 \\
&\quad = \int_{R^d} \phi^2(z)\,\mu_{ss}(z)\,dz - \left(\int_{R^d} \phi(z)\,\mu_{ss}(z)\,dz\right)^2 \\
&\qquad + \int_{R^d} \phi^2(z)\,\mu_{tt}(z)\,dz - \left(\int_{R^d} \phi(z)\,\mu_{tt}(z)\,dz\right)^2 \\
&\qquad - 2\int_{R^d} \phi(z)\,\mu_{ss}(z)\,e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z)\,dz \\
&\qquad + 2\int_{R^d} \phi(z)\,\mu_{ss}(z)\,dz\left[\int_{R^d} \mu_{ss}(z)\,e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z)\,dz\right. \\
&\qquad \left. + \int_{R^d\times(s,t]} e^{-\lambda(t-r)}\mathscr{T}_{t-r}\phi(x)\,\mu(dx,dr)\right] \\
&\quad = \int_{R^d} \phi(z)\,\mu_{ss}(z)[\phi(z) - e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z)]\,dz \\
&\qquad + \int_{R^d} \phi(z)[\phi(z)\,\mu_{tt}(z) - \mu_{ss}(z)\,e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z)]\,dz \\
&\qquad + \int_{R^d} \phi(z)\,\mu_{ss}(z)\,dz\left\{\int_{R^d} \mu_{ss}(z)[e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z) - \phi(z)]\,dz\right\} \\
&\qquad + \int_{R^d} \phi(z)\left\{\int_{R^d} [\mu_{ss}(z)\,e^{-\lambda(t-s)}(\mathscr{T}_{t-s}\phi(y))\,\mu_{ss}(y)\right. \\
&\qquad \left. - \mu_{tt}(z)\,\phi(y)\,\mu_{tt}(y)]\,dy\right\}dz \\
&\qquad + 2\int_{R^d} \phi(z)\,\mu_{ss}(z)\,dz \int_{R^d\times(s,t]} e^{-\lambda(t-r)}\mathscr{T}_{t-r}\phi(x)\,\mu(dx,dr).
\end{aligned}
$$

The semigroup $\{e^{-\lambda t}\mathscr{T}_t\}$ is differentiable on the Banach space of real functions on R^d having bounded derivatives up to order 2 with norm $\|\phi\|_2 = \sum_{|\alpha|\leqslant 2} \sup_{x\in R^d} |D^\alpha\phi(x)|$, where $D^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_d)^{\alpha_d}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Hence $|\phi(z) - e^{-\lambda(t-s)}\mathscr{T}_{t-s}\phi(z)| \leqslant M\,\|\phi\|_2(t-s)$, where M is a constant. Using this inequality, adding and subtracting $\mu_{ss}(z)\,\phi(z)$ in the second integral, and $\mu_{ss}(z)\,\phi(y)\,\mu_{tt}(y)$ and $\phi(y)\,\mu_{ss}(y)$ in the fourth integral, we have

$$
\begin{aligned}
&E[\langle X_t, \phi\rangle - \langle X_s, \phi\rangle]^2 \\
&\quad \leqslant M_1 \|\phi\|_2^2 (t-s) + \int_{R^d} \phi^2(z)[\mu_{tt}(z) - \mu_{ss}(z)]\, dz \\
&\qquad + \int_{R^d} \phi(z)\, \mu_{ss}(z)\, dz \int_{R^d} \phi(y)[\mu_{ss}(y) - \mu_{tt}(y)]\, dy \\
&\qquad + \int_{R^d} \phi(z)[\mu_{ss}(z) - \mu_{tt}(z)]\, dz \int_{R^d} \phi(y)\, \mu_{tt}(y)\, dy \\
&\qquad + 2 \int_{R^d} \phi(z)\, \mu_{ss}(z)\, dz \int_{R^d \times (s,t]} e^{-\lambda(t-r)} \mathscr{T}_{t-r}\phi(x)\, \mu(dx, dr) \\
&\quad \leqslant M_2 \|\phi\|_2^2 \left\{ (t-s) + \left| \int_{R^d} [\mu_{ss}(z) - \mu_{tt}(z)]\, dz \right| \right. \\
&\qquad \left. + \int_{R^d \times (s,t]} e^{-\lambda(t-r)}\, \mu(dx, dr) \right\} \\
&\quad \leqslant M_2 \|\phi\|_2^2 \left\{ (t-s) + 2 \int_{R^d \times (s,t]} e^{-\lambda(t-r)} \mu(dx, dr) \right\} \\
&\quad \leqslant M_3 \|\phi\|_2^2 (t-s),
\end{aligned}
$$

where we have used our assumption on the smoothness of μ and M_1, M_2, M_3 are constants. Hence the Dudley–Fernique condition is satisfied.

By the linearity of X^n and X, proving convergence of the finite-dimensional distributions is equivalent to showing that for all $m \geqslant 1$, $t_1 \leqslant \cdots \leqslant t_m$, and $\phi_1, ..., \phi_m \in \mathscr{S}(R^d)$, $\sum_{i=1}^m \langle X^n_{t_i}, \phi_i\rangle$ converges weakly to $\sum_{i=1}^m \langle X_{t_i}, \phi_i\rangle$ as $n \to \infty$.

By (2.2) we have

$$
\begin{aligned}
\sum_{i=1}^m \langle X^n_{t_i}, \phi_i\rangle &= n^{-1/2} \sum_{i=1}^m \sum_{k=1}^n [\phi_i(B^k_{t_i}) - E\phi_i(B^k_{t_i})] \\
&= n^{-1/2} \sum_{k=1}^n \sum_{i=1}^m [\phi_i(B^k_{t_i}) - E\phi_i(B^k_{t_i})].
\end{aligned}
$$

Applying the central limit theorem we obtain the convergence of the finite-dimensional distributions.

It remains to show tightness in $D([0, T_0), \mathscr{S}'(R^d))$ and in $D([T_0, \infty), \mathscr{S}'(R^d))$. For this observe that the process X^n can be written as

$$X^n = X^{(1)n} + X^{(2)n},$$

where $X^{(1)n}$ and $X^{(2)n}$ are the fluctuation processes of the empirical distributions of the processes $B^{(1)k}$ and $B^{(2)k}$, $k \geqslant 1$, respectively, defined by

$$B_t^{(1)k} = \begin{cases} \gamma, & \text{if } \ t < \tau^k, \\ Z^k + W^k_{t-\tau^k}, & \text{if } \ t \geqslant \tau^k, \end{cases}$$

and

$$B_t^{(2)k} = \begin{cases} \gamma, & \text{if } \ t < \theta^k \\ \tilde{Z}^k + \tilde{W}^k_{t-\theta^k}, & \text{if } \ t \geqslant \theta^k, \end{cases}$$

where $\tilde{Z}^k = Z^k + W^k_{\theta^k - \tau^k}$ and $\tilde{W}^k_t = W^k_{t+(\theta^k - \tau^k)} - W^k_{\theta^k - \tau^k}$ for $k \geqslant 1$.

By Theorem 2.2 of [6], $X^{(1)n}$ converges weakly in $D([0, T_0), \mathscr{S}'(R^d))$ and in $D([T_0, \infty), \mathscr{S}'(R^d))$ and $X^{(2)n}$ converges weakly in $D([0, \infty), \mathscr{S}'(R^d))$; hence $\{(X^{(1)n}, X^{(2)n})\}_{n \geqslant 1}$ is tight in $D([0, T_0), \mathscr{S}'(R^d)) \times D([0, T_0), \mathscr{S}'(R^d))$ and in $D([T_0, \infty), \mathscr{S}'(R^d)) \times D([T_0, \infty), \mathscr{S}'(R^d)$. Since the limit processes of $X^{(1)n}$ and $X^{(2)n}$ are continuous [6], it then follows (e.g., [1]) that $\{X^{(1)n} + X^{(2)n}\}_{n \geqslant 1}$ is tight in $D([0, T_0), \mathscr{S}'(R^d))$ and in $D([T_0, \infty), \mathscr{S}'(R^d))$, and the proof is finished.

Observe that X can be considered as an $(\mathscr{S}(R^d) \oplus R)'$-valued process, and its covariance functional is given by the right-hand side of (2.4) with $n = 1$ for $\phi \in \mathscr{S}(R^d) \oplus R$.

For the results on the Markov property and the Langevin equation we consider only $\lambda > 0$, since the case $\lambda = 0$ was discussed in [6].

THEOREM 2.2. (i) *The process* $X = \{\langle X_t, \phi \rangle, t \geqslant 0, \ \phi \in \mathscr{S}(R^d)\}$ *is Markovian if and only if* $\mu(R^d \times \{0\}) = 1$.

(ii) *The process* $X^{(a)} \equiv \{\langle X_{a+t}, \phi \rangle, \ t \in [0, b-a], \ \phi \in \mathscr{S}(R^d)\}$ *is Markovian if and only if* $\mu(R^d \times (a, b]) = 0$.

In both (i) *and* (ii) *the Markov property holds strictly.*

(iii) *If* $\mu(R^d \times (a, b]) = 0$, *the process* $X^{(a)}$ *satisfies the Langevin equation*

$$dX_t^{(a)} = (\tfrac{1}{2}\Delta - \lambda)\, X_t^{(a)}\, dt + dW_t, \qquad 0 \leqslant t \leqslant b - a, \tag{2.5}$$

where $W = \{W_t, t \geqslant 0\}$ *is an* $\mathscr{S}'(R^d)$*-valued Wiener process with covariance functional*

$$K_W(s, \phi; t, \psi) = \int_0^{s \wedge t} \langle Q_u \phi, \psi \rangle \, du, \qquad \phi, \psi \in \mathscr{S}(R^d), \tag{2.6}$$

with

$$\langle Q_u \phi, \psi \rangle = \int_{R^d} \mu_{a+u, a+u}(x) [\nabla \phi(x) \cdot \nabla \psi(x) + \lambda \phi(x)\, \psi(x)] \, dx \tag{2.7}$$

($\cdot$ *denotes the scalar product in* R^d). *In particular, if* $\mu(R^d \times \{0\}) = 1$ *the process* X *satisfies* (2.5)–(2.7) *with* $a = 0$, $b = \infty$.

Proof. The Markov property holds if and only if given $s < t$, there exists an $\mathcal{S}'(R^d)$-valued random variable $X_{s,t}$ which satisfies condition (M). Observe that given $r \leqslant s \leqslant t$, $\phi, \psi \in \mathcal{S}(R^d)$, from (2.4) we have

$$\begin{aligned}
&\operatorname{Cov}[\langle X_r, \phi\rangle, \langle X_t, \psi\rangle] \\
&\quad = \operatorname{Cov}[\langle X_r, \phi\rangle, \langle X_s, e^{-\lambda(t-s)}\mathcal{T}_{t-s}\psi\rangle] \\
&\qquad - \int_{R^d} \phi(y)\, \mu_{rr}(y)\, dy \int_{R^d \times (s,t]} e^{-\lambda(t-v)}\mathcal{T}_{t-v}\psi(x)\, \mu(dx, dv). \quad (2.8)
\end{aligned}$$

Suppose first that $X_{s,t}$ is of the form $\langle X_{s,t}, \psi\rangle = \langle X_s, e^{-\lambda(t-s)}\mathcal{T}_{t-s}\psi - C_\psi\rangle$, where C_ψ is a constant function which depends on ψ (recall that the process X is well-defined for $\phi \in \mathcal{S}(R^d) \oplus R$). Clearly $\langle X_s, \mathcal{S}(R^d) \oplus R\rangle \subset \langle X_s, \mathcal{S}(R^d)\rangle^-$. By (2.8), in order to have the Markov property it suffices to prove

$$\begin{aligned}
&\operatorname{Cov}[\langle X_r, \phi\rangle, \langle X_s, C_\psi\rangle] \\
&\quad = \int_{R^d} \phi(y)\, \mu_{rr}(y)\, dy \int_{R^d \times (s,t]} e^{-\lambda(t-v)}\mathcal{T}_{t-v}\psi(x)\, \mu(dx, dv). \quad (2.9)
\end{aligned}$$

By (2.4),

$$\begin{aligned}
&\operatorname{Cov}[\langle X_r, \phi\rangle, \langle X_s, C_\psi\rangle] \\
&\quad = C_\psi\, e^{-\lambda(s-r)} \int_{R^d} \phi(z)\, \mu_{rr}(z)\, dz \\
&\qquad - C_\psi \int_{R^d} \phi(y)\, \mu_{rr}(y)\, dy \left[\int_{R^d} \mu_{rs}(y)\, dy + \int_{R^d \times (r,s]} e^{-\lambda(s-v)}\mu(dx, dv)\right] \\
&\quad = C_\psi (e^{-\lambda(s-r)} - f_{ss}(1)) \int_{R^d} \phi(z)\, \mu_{rr}(z)\, dz. \quad (2.10)
\end{aligned}$$

From (2.9) and (2.10) we obtain

$$C_\psi = (e^{-\lambda(s-r)} - f_{ss}(1))^{-1} \int_{R^d \times (s,t]} e^{-\lambda(t-v)}\mathcal{T}_{t-v}\psi(x)\, \mu(dx, dv).$$

For condition (M) to hold C_ψ must not depend on r, and since $\lambda > 0$, this occurs if and only if $\mu(R^d \times (s, t]) = 0$, and since s and t are arbitrary, (M) holds if and only if $\mu(R^d \times \{0\}) = 1$.

Suppose now that $\langle X_{s,t}, \psi \rangle \in \langle X_s, \mathscr{S}(R^d) \rangle^-$ satisfies (M). From our previous calculations we see that provided that we restrict ourselves to $r=s$, then (M) holds for

$$\langle X_{s,t}, \psi \rangle \equiv \langle X_s, \bar{\psi} \rangle \tag{2.11}$$

with

$$\bar{\psi} = e^{-\lambda(t-s)} \mathscr{T}_{t-s} \psi$$
$$- (1 - f_{ss}(1))^{-1} \int_{R^d \times (s,t]} e^{-\lambda(t-v)} \mathscr{T}_{t-v} \psi(x)\, \mu(dx, dv) \quad (\bar{\psi} \in \mathscr{S}(R^d) \oplus R).$$

We will show that the validity of (M) for $r=s$ implies that (2.11) holds in general.

By assumption there exists a sequence $\{\phi_n\}_n$ in $\mathscr{S}(R^d)$ such that $\langle X_s, \phi_n \rangle$ converges to $\langle X_{s,t}, \psi \rangle$ in L^2 as $n \to \infty$. Then, using (M) with $r=s$ we have

$$\begin{aligned}
E\langle X_s, \bar{\psi} \rangle^2 &= E\langle X_s, \bar{\psi} \rangle \langle X_t, \psi \rangle \\
&= E\langle X_s, \bar{\psi} \rangle \langle X_{s,t}, \psi \rangle \\
&= \lim_{n \to \infty} E\langle X_s, \bar{\psi} \rangle \langle X_s, \phi_n \rangle \\
&= \lim_{n \to \infty} E\langle X_t, \psi \rangle \langle X_s, \phi_n \rangle \\
&= \lim_{n \to \infty} E\langle X_{s,t}, \psi \rangle \langle X_s, \phi_n \rangle \\
&= E\langle X_{s,t}, \psi \rangle^2.
\end{aligned}$$

Therefore, again by (M) with $r=s$,

$$\begin{aligned}
E|\langle X_{s,t}, \psi \rangle - \langle X_s, \bar{\psi} \rangle|^2 &= E\langle X_{s,t}, \psi \rangle^2 + E\langle X_s, \bar{\psi} \rangle^2 - 2E\langle X_{s,t}, \psi \rangle \langle X_s, \bar{\psi} \rangle \\
&= E\langle X_{s,t}, \psi \rangle^2 + E\langle X_s, \bar{\psi} \rangle^2 - 2E\langle X_s, \bar{\psi} \rangle^2 \\
&= E\langle X_{s,t}, \psi \rangle^2 - E\langle X_s, \bar{\psi} \rangle^2 \\
&= 0,
\end{aligned}$$

and hence $\langle X_{s,t}, \psi \rangle = \langle X_s, \bar{\psi} \rangle$ a.s.

But we have seen in the first part of the proof that if

$$\langle X_{s,t}, \psi \rangle = \langle X_s, e^{-\lambda(t-s)} \mathscr{T}_{t-s} \psi + C_\psi \rangle,$$

where C_ψ is a constant, then $\mu(R^d \times \{0\}) = 1$.

To obtain the Langevin equation in (iii) we observe that condition

(SM′) holds for $\mathscr{U}_{s,t} = e^{-\lambda(t-s)}\mathscr{T}_{t-s}$. Then we can apply Theorem 2 of [2] (since $\mathscr{S}(R^d) \oplus R$ is a nuclear Fréchet space), which yields the result.

Remarks. (1) In order to compare the present results for $\lambda > 0$ with those for the case $\lambda = 0$ we recall the latter [6]. In this case the fluctuation limit process is always Markovian. If $\mu(R^d \times \{0\}) = 1$, it is strictly Markovian, and it is strictly Markovian on $[a, b]$ if and only if $\mu(R^d \times (a, b]) = 0$. Thus the new behavior exhibited in the case $\lambda > 0$ is that the possibility of death prevents the Markov property. As we have noted in the Introduction, this is a natural result.

(2) The reason for considering convergence on the two spaces $D([0, T_0), \mathscr{S}'(R^d))$ and $D([T_0, \infty), \mathscr{S}'(R^d))$, rather than on $D([0, \infty), \mathscr{S}'(R^d))$, is the lack of a closed martingale on $[0, T_0]$ for the tightness proof [6].

(3) If $\mu(R^d \times (a, b]) = 0$, $X^{(a)}$ is explicitly written as the unique solution of (2.5):

$$\langle X_t^{(a)}, \phi \rangle = e^{-\lambda t} \langle X_0^{(a)}, \mathscr{T}_t \phi \rangle + \int_0^t e^{-\lambda(t-s)} \langle dW_s, \mathscr{T}_{t-s} \phi \rangle, \qquad 0 \leqslant t \leqslant b-a, \phi \in \mathscr{S}(R^d).$$

3. Related Developments

In the case $\mu(R^d \times \{0\}) = 1$ and $\lambda = 0$, there is an extensive literature on models of the type we have discussed above allowing interactions between the random motions. A small list of recent relevant papers is the following: on fluctuations limits, Dittrich [4], Mitoma [9], Shiga and Tanaka [11], Sznitmann [13], Spohn [12]; on large deviations, Dawson and Gärtner [3], Donsker and Varadhan [5]. It would be interesting to investigate fluctuations and large deviations for such models with space-time creation and/or death ($\mu(R^d \times \{0\}) < 1$, $\lambda > 0$), in particular Markov properties of the fluctuation limits.

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